Improved explanation of the electrocardiogram

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A simple model is presented which is used to calculate the potential outside a nerve or muscle fiber embedded in a uniform conducting medium. The resulting integral is evaluated in the dipole approximation and agrees with equations which have been in the literature for a long time. The result can be used to interpret the electrocardiogram.

I. INTRODUCTION

Several years ago[1,2] I described a model which related the electrocardiogram waveform to the electrical changes taking place in the cells of the heart muscle (myocardium). The model assumed that the source of the electrocardiogram was static charges in a body which consists of empty space. Although some discussion was given of the relation of this model to the actual currents, students still have difficulty believing that the body can be approximated by a vacuum. It is not much more difficult to use a more satisfying model which recognizes the fact that the signal is due to superposition of currents which flow in the body as a result of the electrical changes in each myocardial cell.

The model is based on the calculation of the potential outside a cell which is stretched along the x axis and embedded in an infinite, homogeneous conductor. First, the current from a segment of the cell into the infinite conducting medium is related to the potential distribution inside the cell. Then the potential distribution in the medium is calculated by superposing solutions for current injected into a homogeneous conducting medium at different points along the x axis.

The result agrees with calculations in the literature which are obtained by solving Maxwell's equations with inductive and capacitive effects and propagation times ignored. This approach, by ignoring those effects when the model is constructed, makes the calculation accessible to students who have had only one year of calculus.

II. CURRENT FROM A SEGMENT OF A CELL

A resting nerve or muscle cell has an interior potential which is about 80 mV less than outside. The electric field across the cell membrane is generated by a layer of negative charge on the inside and a positive layer of charge on the outside of the membrane. As the nerve cell conducts or the muscle cell prepares to contract, an electrochemical wave of depolarization and subsequent repolarization travels along the cell.[2,3]

Consider a single cell stretched along the x axis. The current density inside can be assumed to be uniform,[4] with total current $i_i(x,t)$ flowing in the x direction. Figure 1 shows a segment of the cell between x and $x+dx$. The current into the segment across surface A is $i_i(x,t)$; current $i_i(x+dx,t)$ crosses surface A'. The difference between these is a current crossing surface AA'. Part of this current flows through the (shaded) membrane. The remainder charges the membrane capacitance:

$$i_i(x,t) - i_i(x+dx,t) = (2\pi a\, dx)\left[c_m\left(\frac{\partial v}{\partial t}\right) + j_m\right]. \qquad (1)$$

Here a is the cell radius, c_m the membrane capacitance per unit area, v the potential across the cell membrane, and j_m the membrane current density. Note that surface AA' is just inside the membrane. The two layers of polarization charge are between AA' and BB'. A current di_0 out through surface BB' is made up of the membrane charging (displacement) current and j_m.

$$di_0 = -\left(\frac{\partial i_i(x,t)}{\partial x}\right) dx. \qquad (2)$$

The potential across the membrane is $v(x,t) = v_i(x,t) - v_0(x,a,t)$, where $v_0(x,a,t)$ is measured just outside the membrane at radius a. Because the cell contents obey Ohm's law, the interior current is

$$i_i(x,t) = -\pi a^2 \sigma_i\left(\frac{\partial v_i}{\partial x}\right), \qquad (3)$$

where σ_i is the conductivity of the material in the cell. The exterior current is

$$di_0 = \pi a^2 \sigma_i\left(\frac{\partial^2 v_i}{\partial x^2}\right) dx. \qquad (4)$$

III. POTENTIAL OUTSIDE THE CELL

Consider now the current in the external medium. Imagine that the radius of the cell is so small that the influence of the cell can be replaced by a current distribution $di_0(x)$ along the x axis.

If current di_0 is injected at the origin into an infinite homogeneous conductor of conductivity σ_0, the current in the medium will be directed radially outward and will have spherical symmetry. The current density at distance r will be $j = di_0/4\pi r^2$, the electric field will be j/σ_0, and the potential (with $v=0$ taken at infinity) will be

$$v(r) = di_0/4\pi\sigma_0 r. \qquad (5)$$

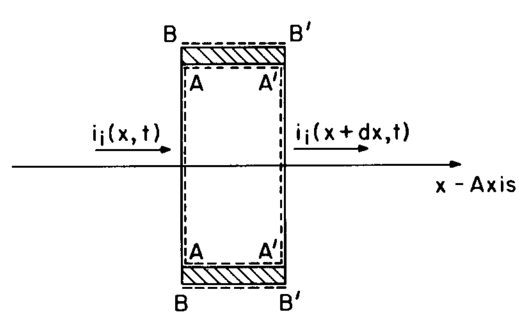

Fig. 1. A portion of a cell stretched along the x axis is shown. It is sliced parallel to the axes of the cylinder, with the cut portions of the membrane shaded. An interior current $i_i(t)$ flows along the axis.

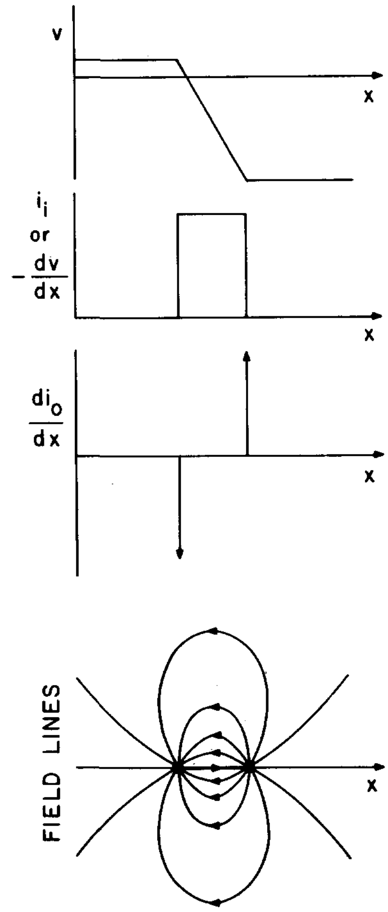

Fig. 2. The potential, current in the cell, current per unit length injected into the external medium, and lines of electric field and current flow in the external medium are shown. For typical cells, the transition takes place in from 1 to 5 mm.

For a distribution of currents $di_0(x)$ along the x axis, the exterior potential at a point far from the cell can be obtained by superposition:

$$v_{\text{ext}} = \int dv_{\text{ext}} = \int \frac{di_0}{4\pi\sigma_0 r}. \tag{6}$$

Using Eq. (4), the potential at a point r_0 is

$$v_{\text{ext}}(r_0) = \frac{\pi a^2 \sigma_i}{4\pi\sigma_0} \int \frac{1}{r} \left(\frac{\partial^2 v_i}{\partial x^2} \right) dx. \tag{7}$$

Since r depends on both x and r_0, the coordinate of the observation point, this integral is evaluated by expanding $1/r$ [e.g., Ref. 2, Eq. (7.2)] and keeping only the first-order (dipole) term:

$$\frac{1}{r} \approx \frac{1}{r_0} \left(1 + \frac{x}{r_0} \cos\theta \right). \tag{8}$$

When this is inserted in Eq. (7), the result is

$$v_{\text{ext}}(r_0) = \frac{\pi a^2 \sigma_i}{4\pi r_0 \sigma_0} \int_{x_1}^{x_2} \left[\left(\frac{\partial^2 v_i}{\partial x^2} \right) + \frac{x \cos\theta}{r_0} \left(\frac{\partial^2 v_i}{\partial x^2} \right) \right] dx. \tag{9}$$

The first term is proportional to $\partial v_i/\partial x$ evaluated at the end points. The second term is integrated by parts. If x_1 and x_2 are taken to be at points where $\partial v_i/\partial x$ is zero, then

$$v_{\text{ext}}(r_0) = \frac{\pi a^2 \cos\theta\,(\sigma_i/\sigma_0)}{4\pi r_0^2} [v(x_1) - v(x_2)]. \tag{10}$$

One can define a vector $\mathbf{p}$ pointing along the cell from x_1 to x_2 with magnitude

$$p = \pi a^2 (\sigma_i/\sigma_0)[v(x_1) - v(x_2)] \tag{11}$$

and write the exterior potential as

$$v_{\text{ext}}(\mathbf{r}_0) = \mathbf{p} \cdot \mathbf{r}_0 / 4\pi r_0^3. \tag{12}$$

This has the same form as the potential due to a dipole. Vector $\mathbf{p}$ is called the *electric force vector* by physiologists. "Electrical activity" has the advantage of not suggesting that it is a force. Figure 2 shows the electric field lines outside a cell. One can see that they are the same as those due to a dipole.

The region on the cell where the potential changes often occupies only a small region along the x axis. Therefore we can also define $\Delta\Omega = \pi a^2 \cos\theta / r_0^2$ to be the solid angle of a cross section of the cell where the potential changes, subtended at the point where v_{ext} is measured and write

$$v_{\text{ext}} = \left(\frac{\Delta\Omega}{4\pi} \right) \left(\frac{\sigma_i}{\sigma_0} \right) [v(x_1) - v(x_2)]. \tag{13}$$

This is the analog of Eq. (7.4) in Ref. 2 or of Eq. (3) or the second equation in the Appendix in Ref. 1, or of Eq. (5.68) in Ref. 5.

IV. PHYSIOLOGICAL EXAMPLES

In nerve and muscle cells, the initial depolarization is accompanied by a potential change from -80 mV to some small positive value. This occurs over 1–5 mm along the cell. A schematic plot of the potential change, the current in the cell, the current through the membrane, and the electric field lines outside the cell is given in Fig. 2. One sees the dipolelike field associated with the advancing wave front. For a nerve cell and most muscle cells, the cell immediately repolarizes, so that there is another vector $\mathbf{p}$ pointing in the opposite direction, right behind the first one. Thus there is no exterior potential in the dipole approximation. For a myocardial cell, the depolarization lasts for ~ 100 ms, and the entire cell is completely depolarized for some time. In that case, $v(x_1)$ is 0, $v(x_2)$ is -80 mV, and a relatively large signal exists. In the heart, many cells depolarize at the same time, with the wave of depolarization sweeping through the myocardium, so that $\Delta\Omega$ becomes the solid angle subtended by the entire wave front.

V. CONCLUSION

The currents which give rise to the electrocardiogram have been described in a model which treats the body as an infinite homogeneous conductor. This model is more satisfying to students than the electrostatic model which was proposed earlier. The discussion in Refs. 1 or 2 can be used with Eq. (11) to explain the details of the electrocardiogram signal.

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