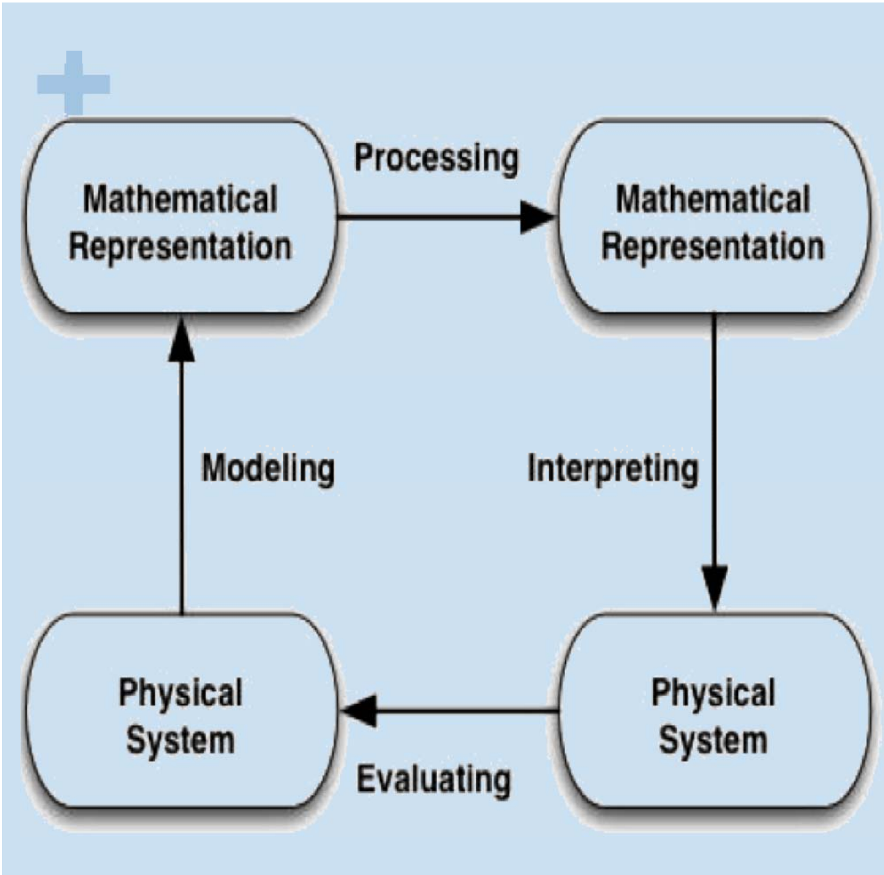

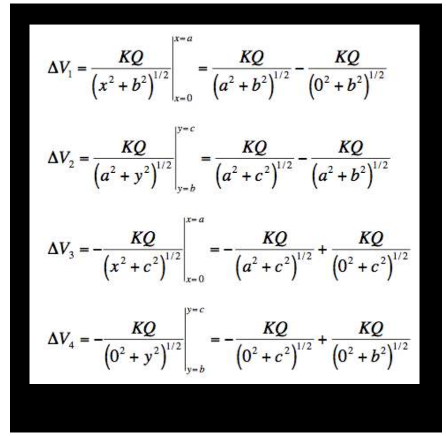

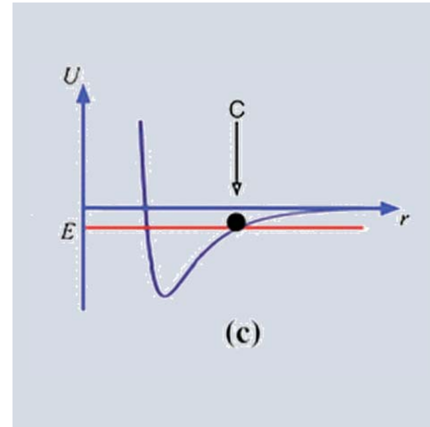

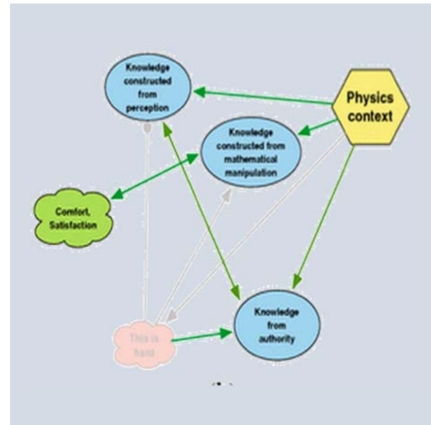

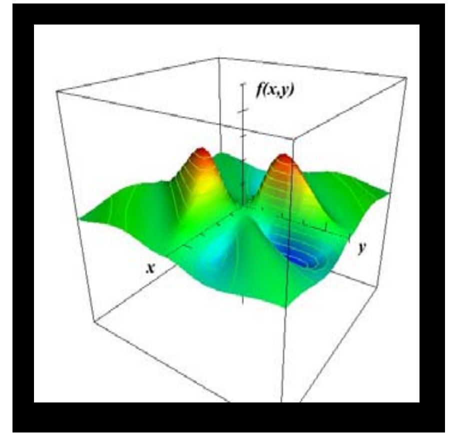

Barriers students face in learning to use math in science

Edward F. Redish
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Problem statement

- Many students who are successful in math classes have difficulties when the same math is used in a science class.
  - In introductory physics classes, biology majors who have earned an A in calculus seem unable to use simple algebra.
  - In advanced physics classes, physics majors who are successful in their math classes may have trouble, make obvious errors, or have trouble making physical sense out of a result.

Math is different in a physics context

In science, we load physical meaning onto symbols

1. Corinne’s shibboleth: separating mathematicians from physicists
2. Filtering an equation through the physics changes how we use it
3. Interpreting equations physically yields hidden functional dependence
4. Adding physical interpretations to symbols lets us hide some very fancy math.

1. Corinne’s Shibboleth*

One of your colleagues is measuring the temperature of a plate of metal placed above an outlet pipe that emits cool air. The result can be well described in Cartesian coordinates by the function

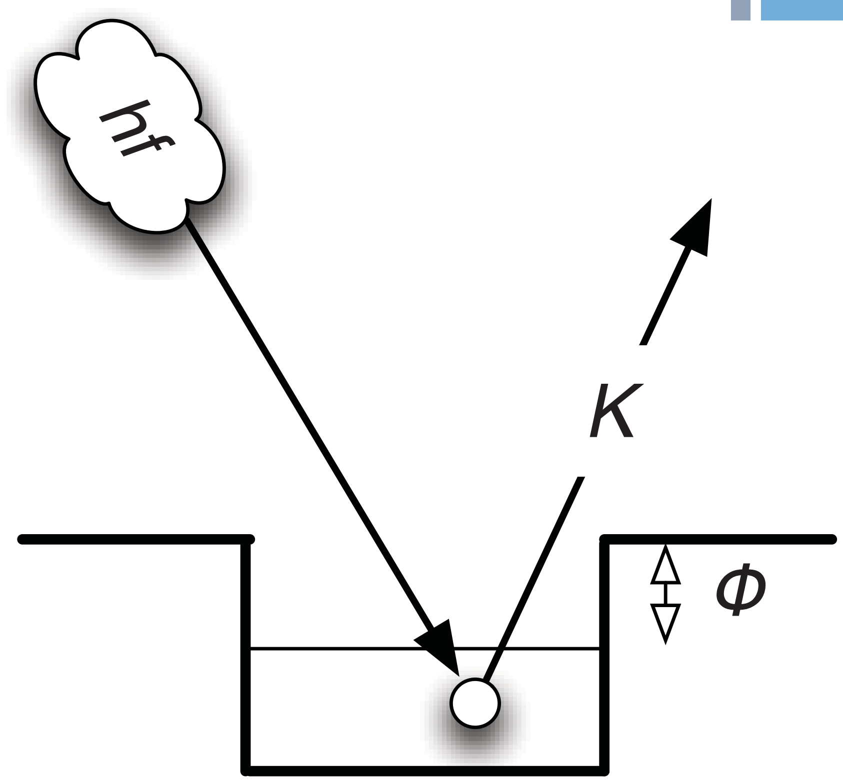
$$T(x,y) = k(x^2 + y^2)$$

where *k* is a constant. If you were asked to give the following function, what would you write?

$$T(r,\theta) = ?$$

+ 2. Filtering an equation through the physics changes the way we use it

If a wavelength of light λ leads to no electrons being emitted at a zero stopping potential, what will happen if we choose a longer wavelength? Explain your reasoning.



+ What's the physics?

- The energy delivered by a photon is hf.
- The frequency of light f is related to its wavelength by $c = f\lambda$.
- The minimum energy needed to extract an electron from a metal is the work function, Φ.
- If a photon is absorbed by an electron in a metal, the maximum KE the electron could have is given by

$$KE = hf - \Phi$$

- KE (which is = ½ mv^2) must be positive so if a photon with energy hf is too small to knock out an electron, so is any photon with a smaller f.

What did the students say?

- In a third semester introductory physics class for engineers, 25% of the students missed this question.
- One wrote:
  - “*If hf – Φ is 0 before the change, a change means that it is no longer 0 so there will be electrons knocked out.*”

The real equation is different from what we write.

It should really have been

$$KE = (hf - \Phi)\theta(hf - \Phi)$$

but we don’t bother putting in the Heaviside function since we “fold the equation through the physics” first.

If there isn’t enough energy to make the KE positive, we don’t use the equation.

3. Interpreting equations physically yields hidden functional dependence

A very small charge q is placed at a point $\vec{r}$ somewhere in space. Hidden in the region are a number of electrical charges. The placing of the charge q does not result in any changes in the position of the hidden charges. The charge q feels a force of magnitude F. We conclude that there is an electric field that the point $\vec{r}$ that has the value $E = F/q$.

If the charge q were replaced by a charge $-3q$, then the electric field at the point $\vec{r}$ would be

a) Equal to $-E$
b) Equal to E
c) Equal to $-E/3$
d) Equal to $E/3$
e) Equal to some other value not given here
f) Cannot be determined from the information give.

+ The force on a charge *q* at the point *r* due to charges Q_1, Q_2, Q_3,...

$$\vec{F} = \frac{kqQ_1}{\left|\vec{r}-\vec{r}_1\right|^3}\left(\vec{r}-\vec{r}_1\right) + \frac{kqQ_1}{\left|\vec{r}-\vec{r}_2\right|^3}\left(\vec{r}-\vec{r}_2\right) + \frac{kqQ_1}{\left|\vec{r}-\vec{r}_3\right|^3}\left(\vec{r}-\vec{r}_3\right) + \ldots$$

$$\frac{\vec{F}}{q} = \frac{kQ_1}{\left|\vec{r}-\vec{r}_1\right|^3}\left(\vec{r}-\vec{r}_1\right) + \frac{kQ_1}{\left|\vec{r}-\vec{r}_2\right|^3}\left(\vec{r}-\vec{r}_2\right) + \frac{kQ_1}{\left|\vec{r}-\vec{r}_3\right|^3}\left(\vec{r}-\vec{r}_3\right) + \ldots = \vec{E}$$

+ What did the students say

A very small charge q is placed at a point $\vec{r}$ somewhere in space. Hidden in the region are a number of electrical charges. The placing of the charge q does not result in any changes in the position of the hidden charges. The charge q feels a force of magnitude F. We conclude that there is an electric field that the point $\vec{r}$ that has the value $E = F/q$.

If the charge q were replaced by a charge -3q, then the electric field at the point $\vec{r}$ would be

a) Equal to -E
b) Equal to E
c) Equal to -E/3
d) Equal to E/3
e) Equal to some other value not given here
f) Cannot be determined from the information give.

More than half of 200 students in an algebra-based physics class gave the answer (c). This is the correct answer if F is interpreted as a fixed symbol (constant) instead of as the force acting on charge q.

+ 4. Adding physical interpretations to symbols lets us hide some fancy math

- Why do we check units?
  - Because we want our equations to hold true whatever units we choose.
  - This is a symmetry principle. It says that a physical equations should transform so that it maintains its equality when something (an arbitrary scale) changes.
- What's the math?
  - In math this says, "Our equations must transform covariantly with respect to transformations of the scaling group SxSxS."

+ How can we get away with this in intro physics?

- We get away with it because we use meaning mapped from everyday experience onto our symbols instead of the math. (Is this the reason so many math instructors refuse to use units?)
- There is a similar situation for vectors (rotational and Euclidean) and vector fields.

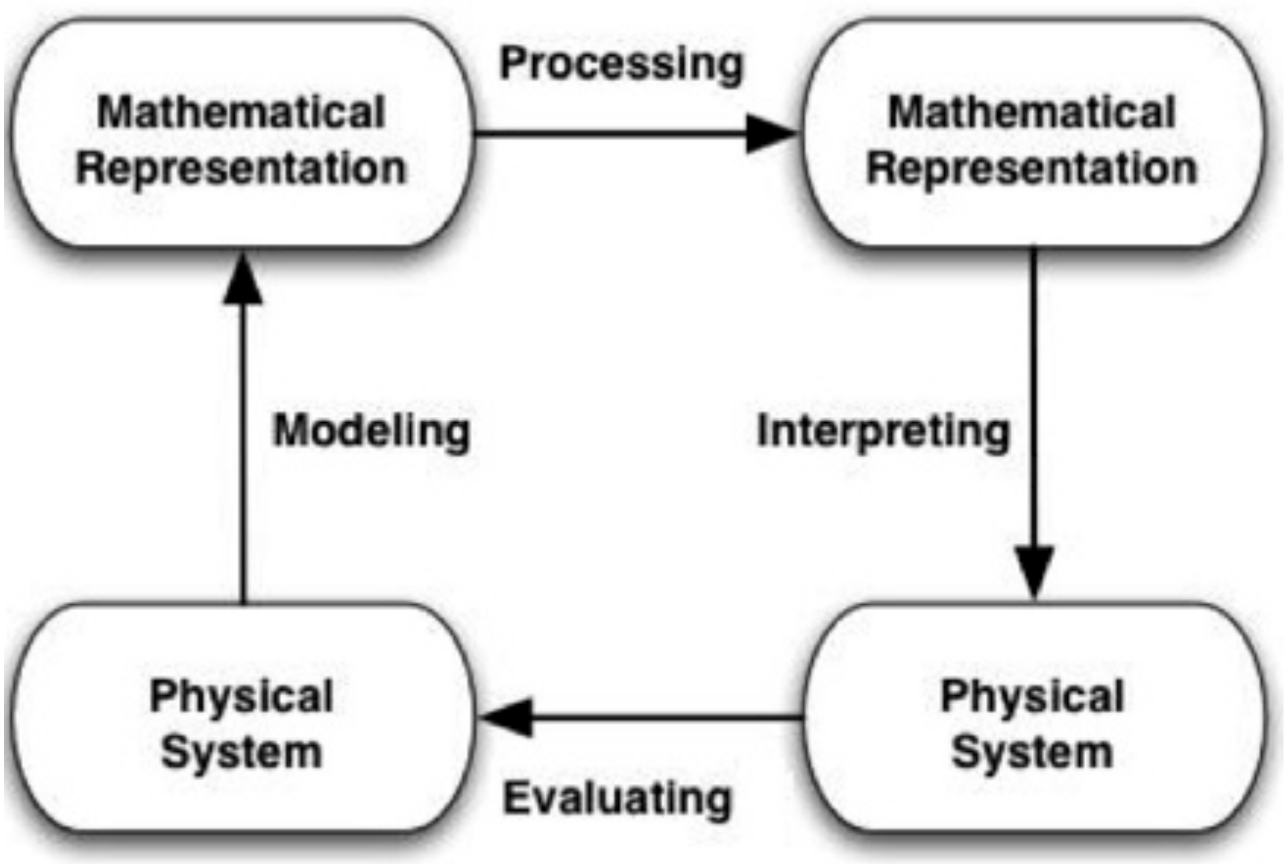
+ And lots more!

- Math in math classes make clean distinctions between constants and variables. In physics we change what we like depending on the situation.
- Math in math classes uses few symbols in sharply defined circumstances. (Variables are almost always x, y, z... Constants are a, b, c...) In physics many symbols are used and are intended to activate physical interpretations.
- Equations in math class rarely have parameters. Equations in physics class always do.

Making meaning with math

Scientists and mathematicians
have different goals for using math

+ In science, math is used to model physical relationships

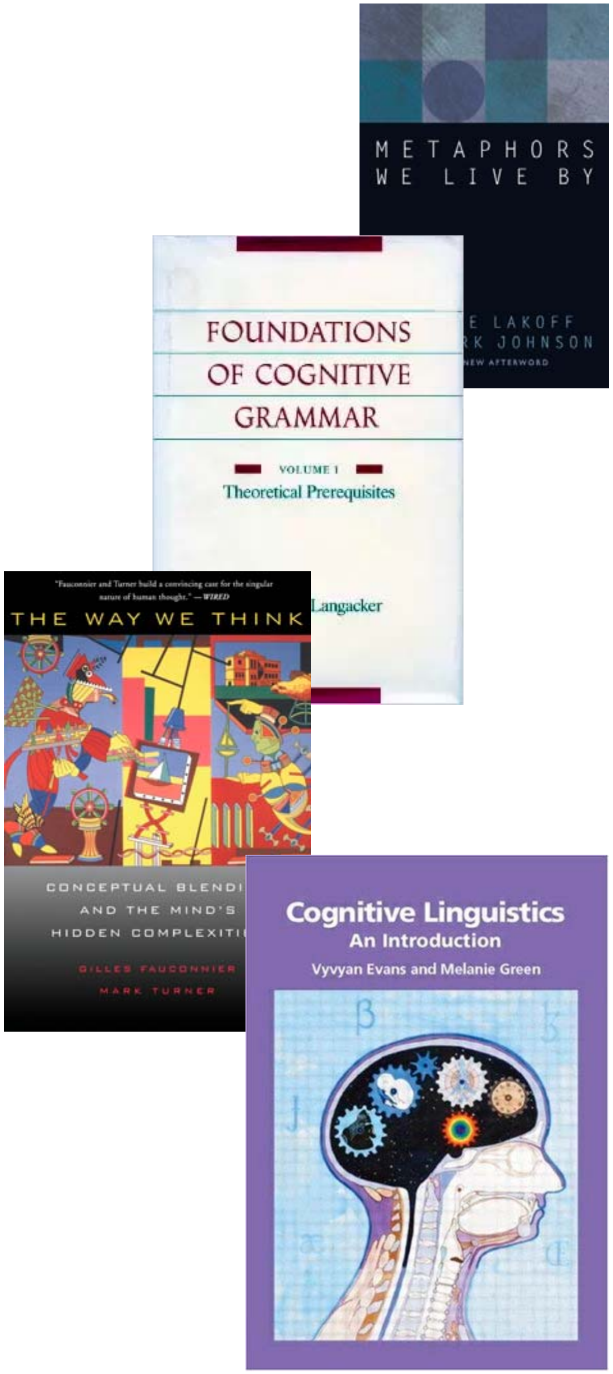


+ Mathematicians and scientists "make meaning" with math in different ways.

- In order to analyze these differences, we need a language for talking about "meaning."
- What does it mean for something to "mean something"?
- For this, we turn to cognitive linguistics, semantics, and pragmatics.

Sources

- Lakoff & Johnson
  Metaphors We Live By
- Langacker
  Foundations of Cognitive Grammar
- Fauconnier & Turner
  The Way we Think
- Evans & Green
  Cognitive Semantics



Three keys to meaning

A. *Embodied cognition:*
Meaning is grounded in physical experience.

B. *Encyclopedic knowledge*:
Ancillary knowledge creates meaning.

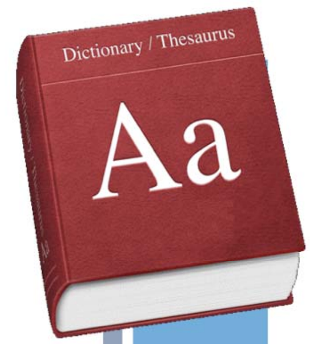
C. *Contextualization*:
Meaning is constructed dynamically
by reference to context.

A. Embodied cognition

- In the end, all of our understanding of even complex concepts must come down to direct perceptual experience.
- Many processes enable the building of this extraordinary and complex linguistic structure:
  - Metaphor (Lakoff & Johnson)
  - Polysemy (Langacker, Evans)
  - Blending (Fauconnier & Turner)

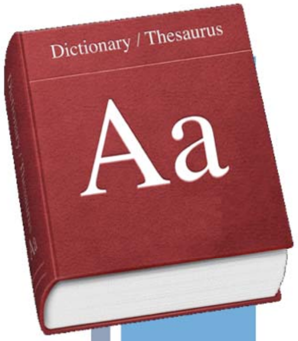
* Note the similarity to the idea of “phenomenological primitives” in the “Knowledge-in-Pieces” framework.

Dictionary meaning?



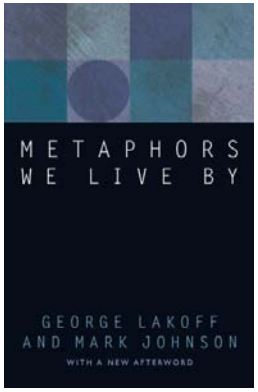
- I do not read Arabic, but I know a bit about how an Arabic dictionary might work.
  - It would read from right to left.
  - The word to be defined would be on the right of an entry.
- From that, I could find the word "الألكترونات" given enough time.
- But finding it would not help me figure out what it means.
- So how *does* a dictionary work?

Dictionary meaning!



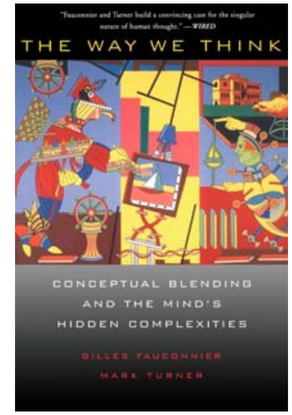
- Dictionaries are fundamentally circular: words are defined in terms of words.
- The value of a dictionary lies in the hope that as you traverse the circle, you will come upon some set of terms that you already know (and have ultimately learned in some other way than from definitions) – for example as an infant from matching words to visual and tactile experience.

Chains of metaphor

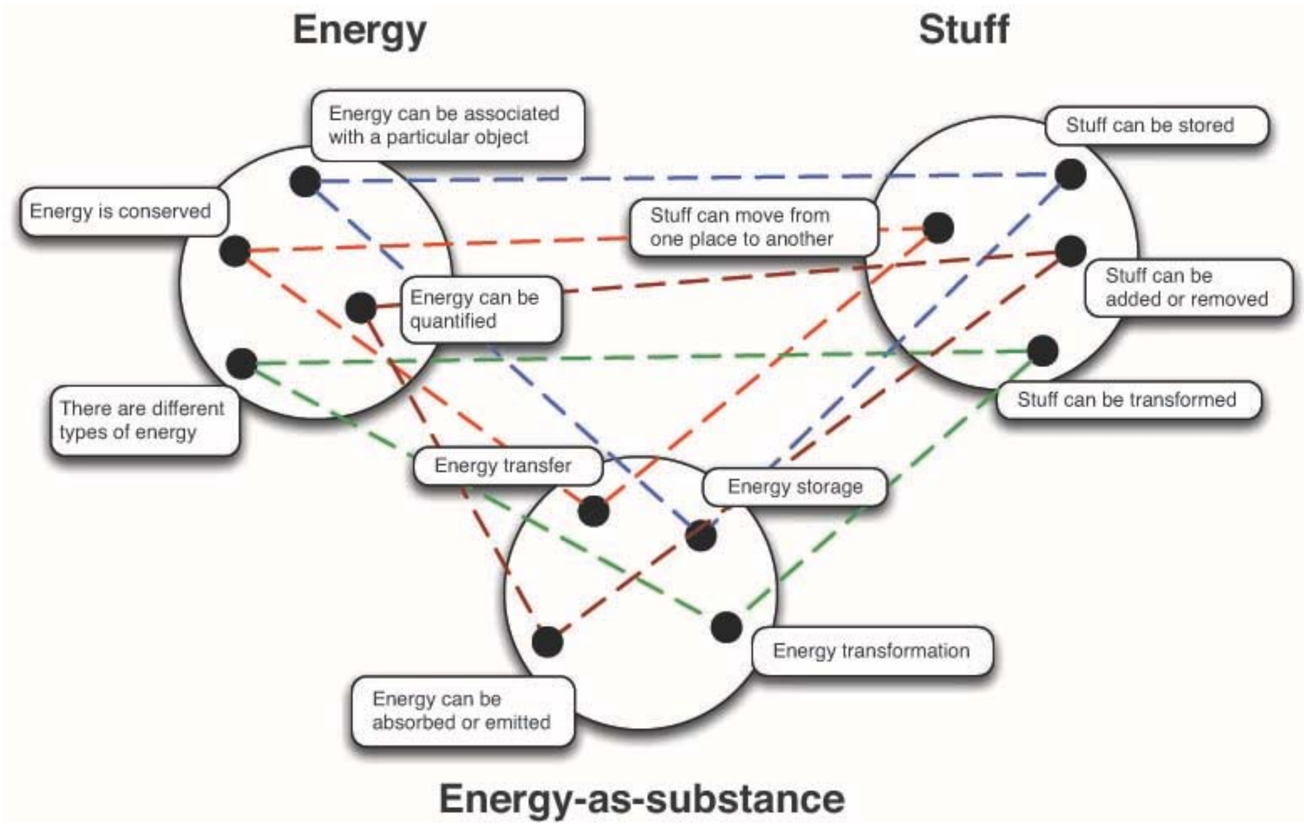


- Lots of other more abstract ideas can be built through metaphors associated with embodied (physical) experience.
  - “The stock market **crashed**.”
  - “I’m so **over** him.”
  - “These days I’ve really gotten **into** studying linguistics.”

+ More complex structures can be built through blending of distinct mental spaces

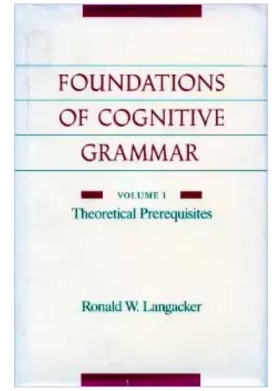


- Generalizing metaphor and analogy.

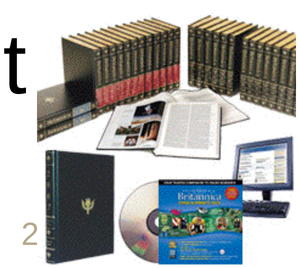


Dreyfus, Gupta & Redish, *Int. J. Sci. Ed.* (2015) in press

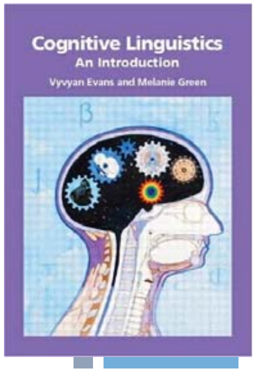
+ B. "Encyclopedic" knowledge is critical to meaning



- We understand the meaning of words not in terms of terse definitions provided in a dictionary but in reference to a contextual web of concepts.
- In a very real sense a "meaning" is the set of associated knowledge that we activate when we hear or interpret something.



C. Contextualization*



- Language does not directly code for semantic meaning.
- Rather, linguistic units are prompts for the construction of meaning within a given conceptual / contextual frame.
- This means that meaning is dynamically constructed – a process rather than something fixed and stable.

* Evans actually uses the term “conceptualization” but I choose to focus on “context” since “concept” has another meaning in PER.

+ Example

- “The child is safe.

  The beach is safe.”

- What do these mean?

Embodied cognition: Math Example – Symbolic Forms

- A symbolic form blends a mathematical *symbol template* with an abstraction of an understanding of relationships obtained from embodied experience.

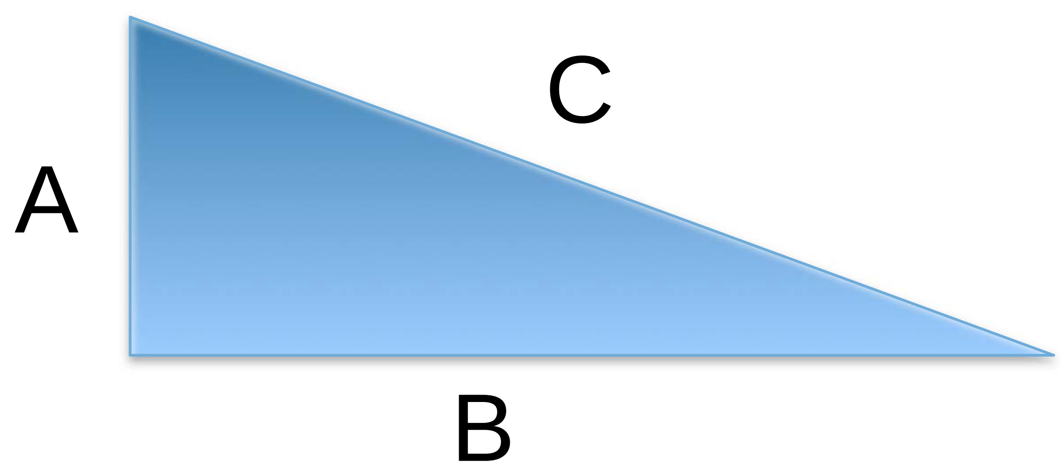
  "Parts of a whole": □ = □ + □ + □ +…

  "Base plus change": □ = □ + Δ

Sherin , *Cog. & Instr.* 19 (2001) 479-541

Encyclopedic knowledge: Math example – Pythagoras

- "The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs."

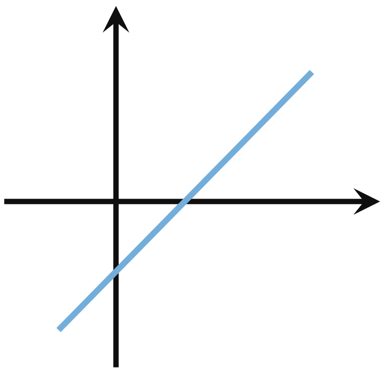


Contextualization: Math example – equation of a line

- An expression containing four symbols and two relational operators

$$y = mx + b$$

- Can be interpreted in the context of a graph is specific associations are made.



Students' can choose
to make meaning with math
in different ways

Making meaning: A physics equation

- The physics equation

$$v = v_0 + at$$

is similar in structure to

$$y = mx + b$$

but can activate different encyclopedic and contextual knowledge.

* Could it be that the different order in the physics form is meant to cue a “base + change” interpretation?

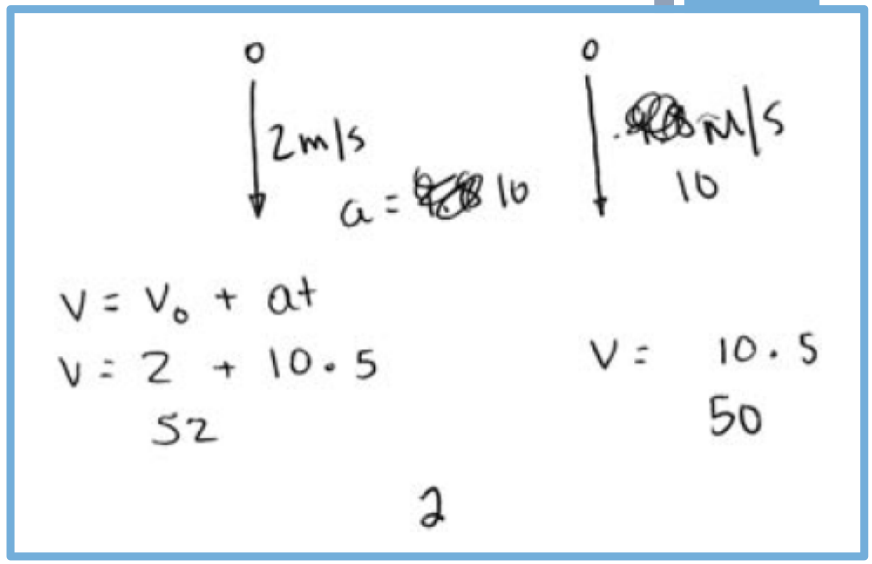
An interview question

- Suppose you are standing with two tennis balls on the balcony of a fourth floor apartment. You throw one ball down with an initial speed of 2 meters per second; at the same moment, you just let go of the other ball, i.e., just let it fall. I would like you to think aloud while figuring out what is the difference in the speeds of the two balls after 5 seconds—is it less than, more than, or equal to 2 meters per second?
- (Acceleration due to gravity is 10 meters per second squared and, if they ask, they are told to neglect air resistance.)

Kuo, Hull, Gupta & Elby , *Sci. Ed* 97:1 (2013) 32-57

One response

- Alex worked out the speeds of the two balls to be 50 and 52 meters per second, then explained her thought process:



: . . . Okay, so after I plug this into the velocity equation, I use the acceleration and the initial velocity that's given, multiply the acceleration by the time that we're looking at, five seconds, and then once I know the velocities after five seconds of each of them, I subtract one from the other and get two. So the question asks "is it more than, less than, or equal to two?" so I would say equal to two

Another response

- Pat also turned to the velocity equation. However, he used it very differently from the way Alex did, focusing on the fact that they change the same way:
- *...the acceleration is a constant and that means that velocity is linearly related to time and they're both at the same, so the first difference is the same. I think it's equal to 2 meters per second.*

They access different resources

- Alex chooses to begin with a full calculation. Pat focuses on “base + change”.
- In later discussion Pat indicates associations with concepts of motion are active as well as ideas of derivatives. Alex can cite the derivative results but doesn’t seem to use them.

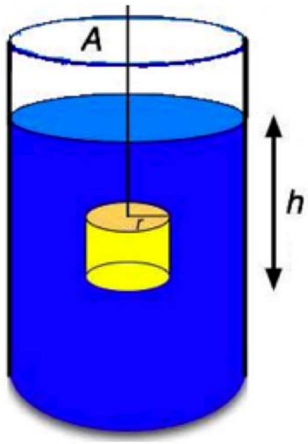
+ Students can take different approaches to problem solving.

- How can we describe the resources that the students have and determine what is going on dynamically as they consider a math problem?
- Our linguistic approach suggests that looking at what kind of knowledge they bring to bear and how they contextualize the problem is important for understanding how they are making meaning.
- This can be both about knowledge (epistemology) and feeling (affect).

The dynamics of making meaning with math

+ Interview with an first year engineering student

- An engineering student is asked about something he hasn't studied yet: pressure under water.



- His intuition is that the pressure increases as you go deeper.
- But when he's given the equation: $P = P_0 + \rho g h$ he gets a sign wrong and concludes the pressure must decrease as you go deeper.

Gupta & Elby , *Int. J. Sci. Ed* 33:18 (2011) 2463-2488

+ Stable under pressure:

Jim: For an equation to be given to you, it has to be like theory and it has to be fact-bearing. So, fact applies for everything. It is like a law. It applies to every single situation you could be in. But, like, your experience at times or perception is just different—or you don't have the knowledge of that course or anything. So, I will go with the people who have done the law and it has worked time after time after time.

I: Do you think the [hydrostatic pressure] equation relates to [the physical experience of pressure]?

Jim: Probably somehow, but not directly. I think there is some way that just completely links the two together, but it's not obvious what that relation is.

+ A dynamic shift:

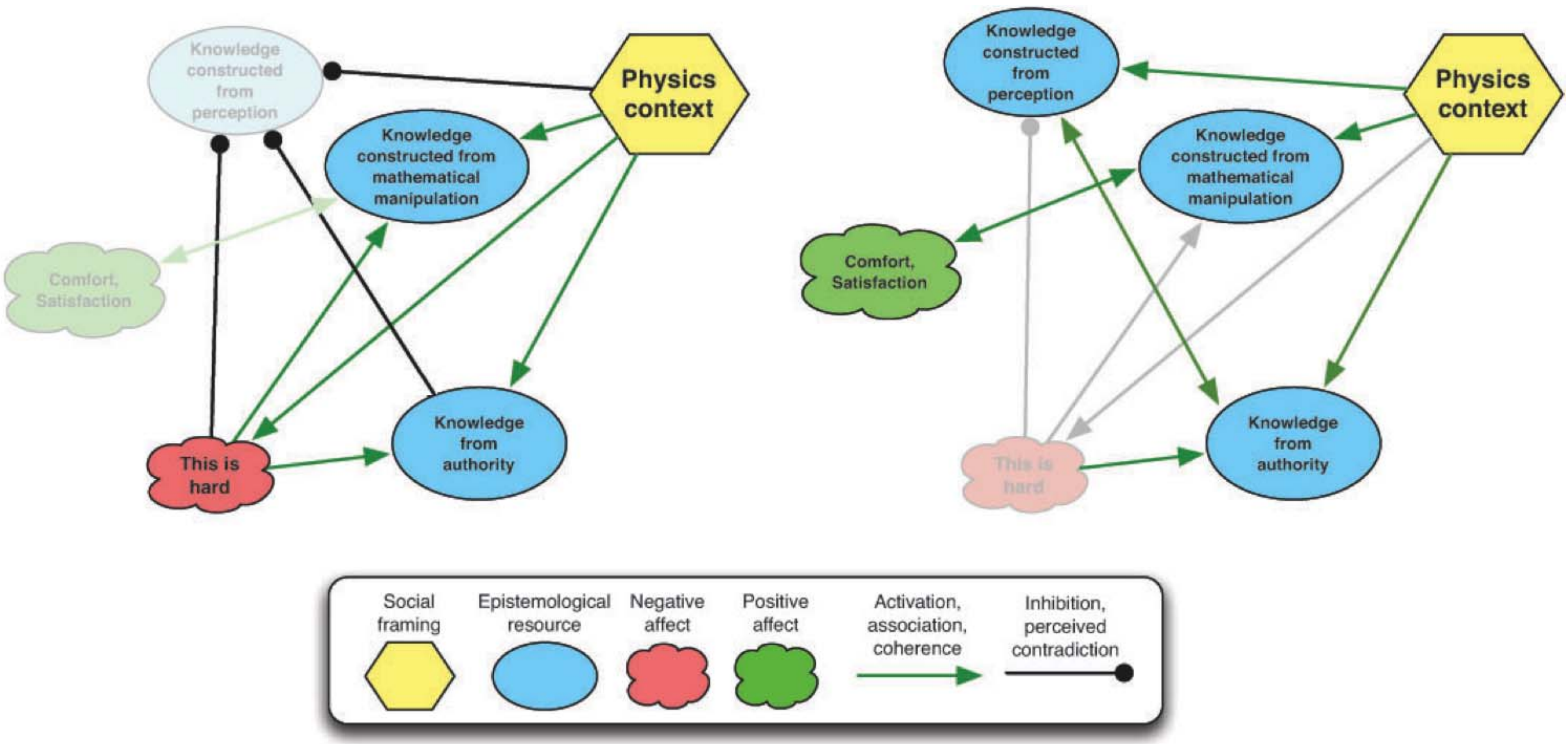
I: What do you think about *g* in the equation? Should that be minus ten or plus ten?

Jim: Oh! minus ten … So, that gives you a positive thing. I would say that the negative does not matter anymore. Oooh! I see. The...lower you go under water the more your pressure is, because the negative and the negative cancel out … So, the more under water you are the higher your pressure is going to be... I forgot to factor in *g*. That's what I think.

I: Okay. Is that more comfortable or less comfortable?

Jim: That is more comfortable because it actually makes more sense to me now. And now your experience actually does work because from your experience being under water you felt more pressure ... If I take into consideration both negatives, it's just positive, they just add up.

+ Activation and inhibition of resources: Interaction of epistemology and affect



Implications for instruction

Teaching physics standing on your head

Can this kind of analysis help?

- This analysis provides a fine-grained approach to understanding student response to the use of math in science.
- That might help us to understand the problems in student-faculty communication and help us to design effective instructional environments.
- Perception of associations and contexts are critical for making meaning. Some tools that have been developed in the ***Resources Framework*** might help describe what is going on.

+ Summary of structures

- Conceptual Resources –
  - Small bits of knowledge, perhaps embodied, perhaps compiled; Associational structure important
- Epistemological Resources –
  - Our “ways of knowing” – affects what resources we choose to bring to bear.
- Affect –
  - Emotional responses, sense of identity – fear, satisfaction, challenge, “I’m a doctor, Jim, not a bricklayer.”
- Expectations –
  - Organized (relatively) stable local coherences learned from previous experiences; can bias what is activated.
- Framing –
  - The process by which the individual’s knowledge of context, social mores and expectations, is matched with their perception of their situation

Example 1: Changing how we talk about equations: Equations as a conceptual organizer

Force is what you have to pay attention to when considering motion

What matters is the sum of the forces

on the object being considered

These relations are independently true for each direction.

$$\vec{a}_A = \vec{F}_A^{net} \Big/ m_A$$

The total force is “shared” to all parts of the object

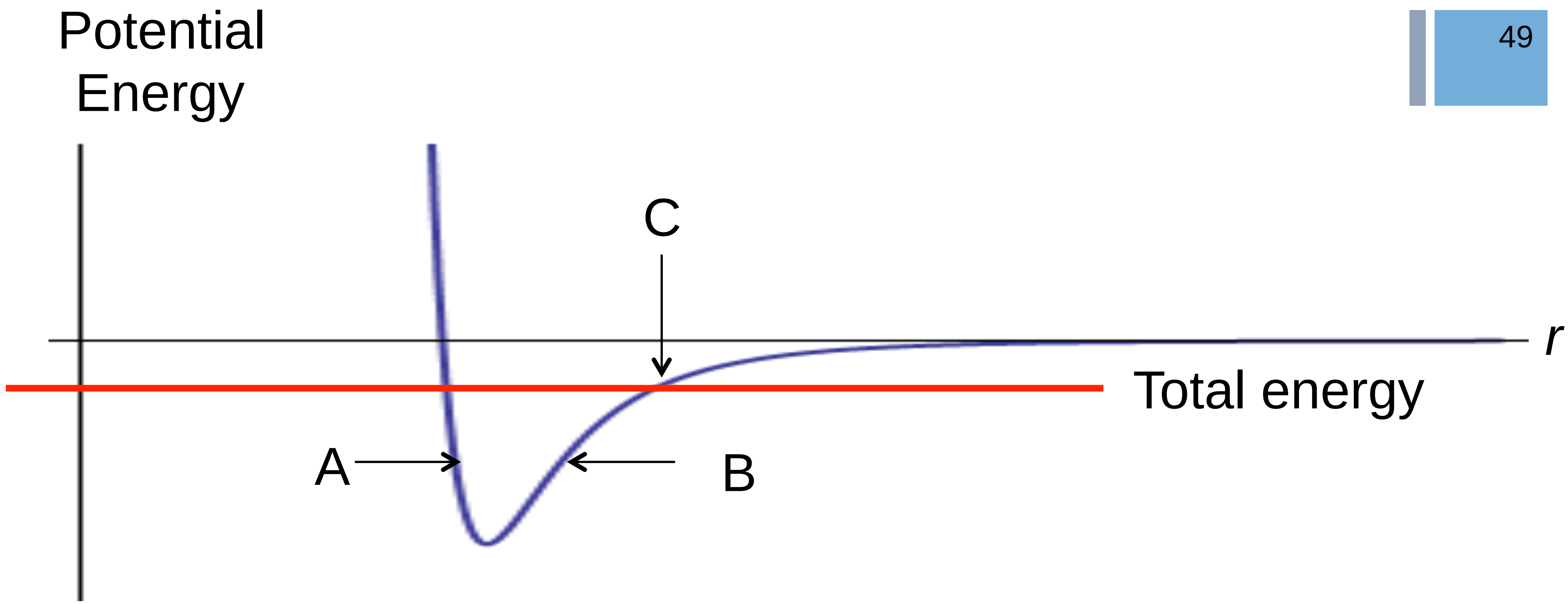
Forces **change** an object’s velocity

You have to pick an object to pay attention to

Total force (shared over the parts of the mass) causes an object’s velocity to change

+ Example 2: Changing how we introduce equations. The “go-to” e-resource

- The ***epistemological stances*** naturally taken by physics instructors and biology students may be dramatically different – even in the context of a physics class.
- One example from my observations of other faculty teaching NEXUS/ Physics yields an insight.

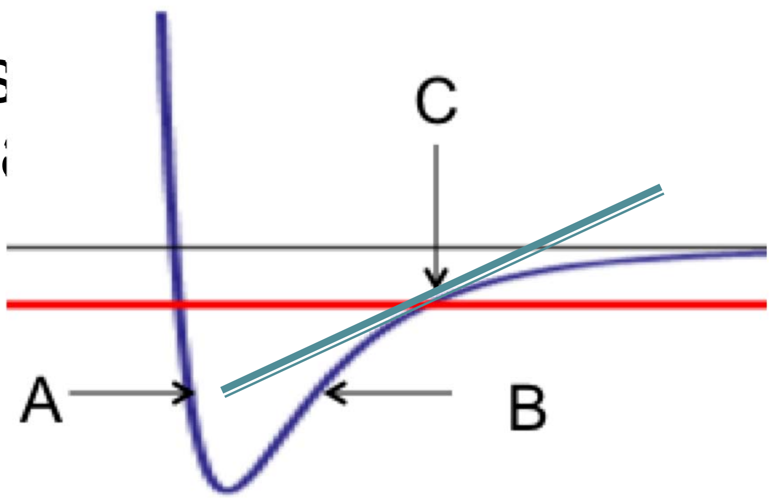


The figure shows the PE of two interacting atoms as a function of their relative separation

Is the **force between the atoms** at the separations marked A,B, and C attractive or repulsive?

+ How two different professors explained when students got stuck.

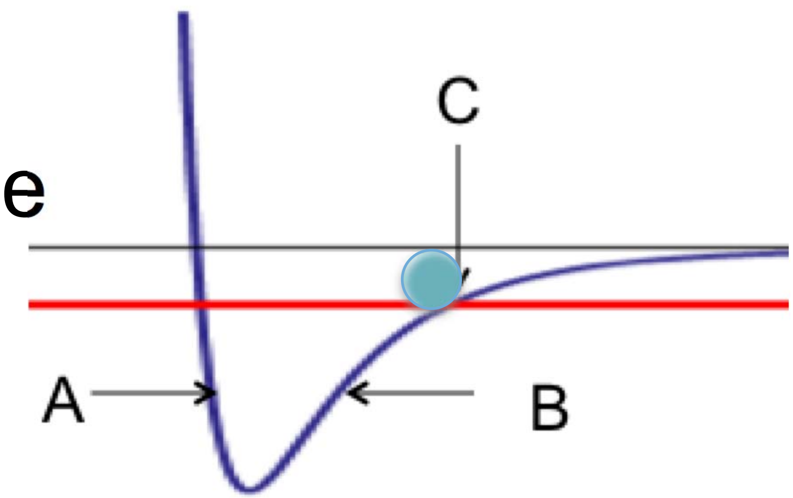
- Remember! $\vec{F} = -\vec{\nabla} U$ (or here) $F = -\frac{dU}{dr}$
- At C, the slope of the *U* graph is positive.
- Therefore the force is negative – to smaller *r*.
- So the potential repres force when the atoms at separation C.



Wandering around the class while students were considering the problem, I found a good response with a different approach.

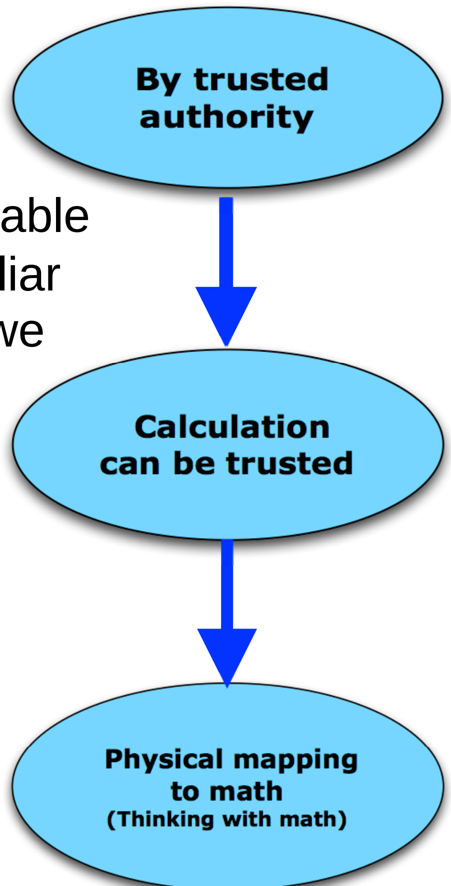
- Think about it as if it were a ball on a hill. Which way would it roll? Why?
- What's the slope at that point?
- What's the force?
- How does this relate to the equation

$$F = -\frac{dU}{dr}$$

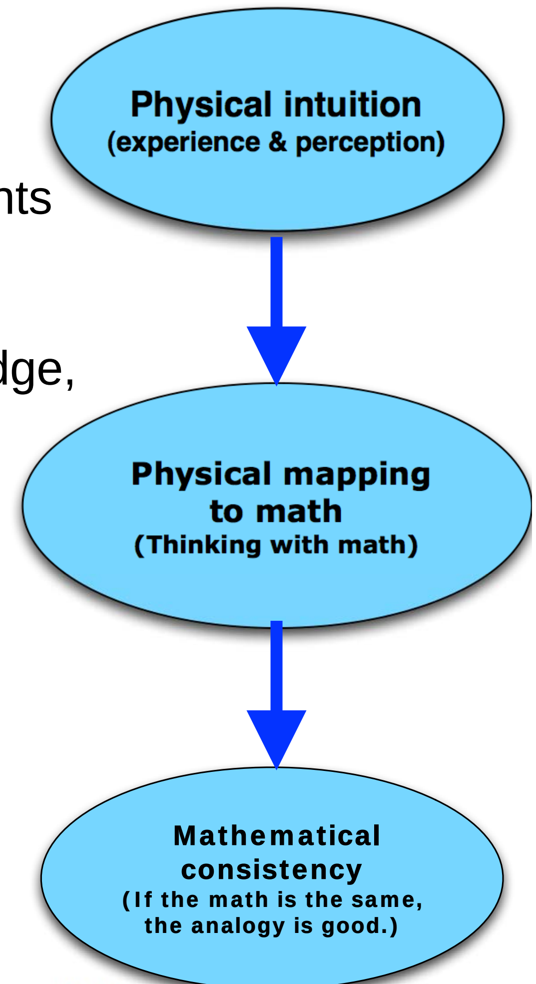


+ I conjecture that a conflict between the epistemological stances of instructor and student make things more difficult.

Physics instructors seem more comfortable beginning with familiar equations – which we use not only to calculate with, but to code and remind us of conceptual knowledge.



Most biology students lack the experience blending math and conceptual knowledge, so they are more comfortable beginning with physical intuitions.



Teaching physics standing on your head

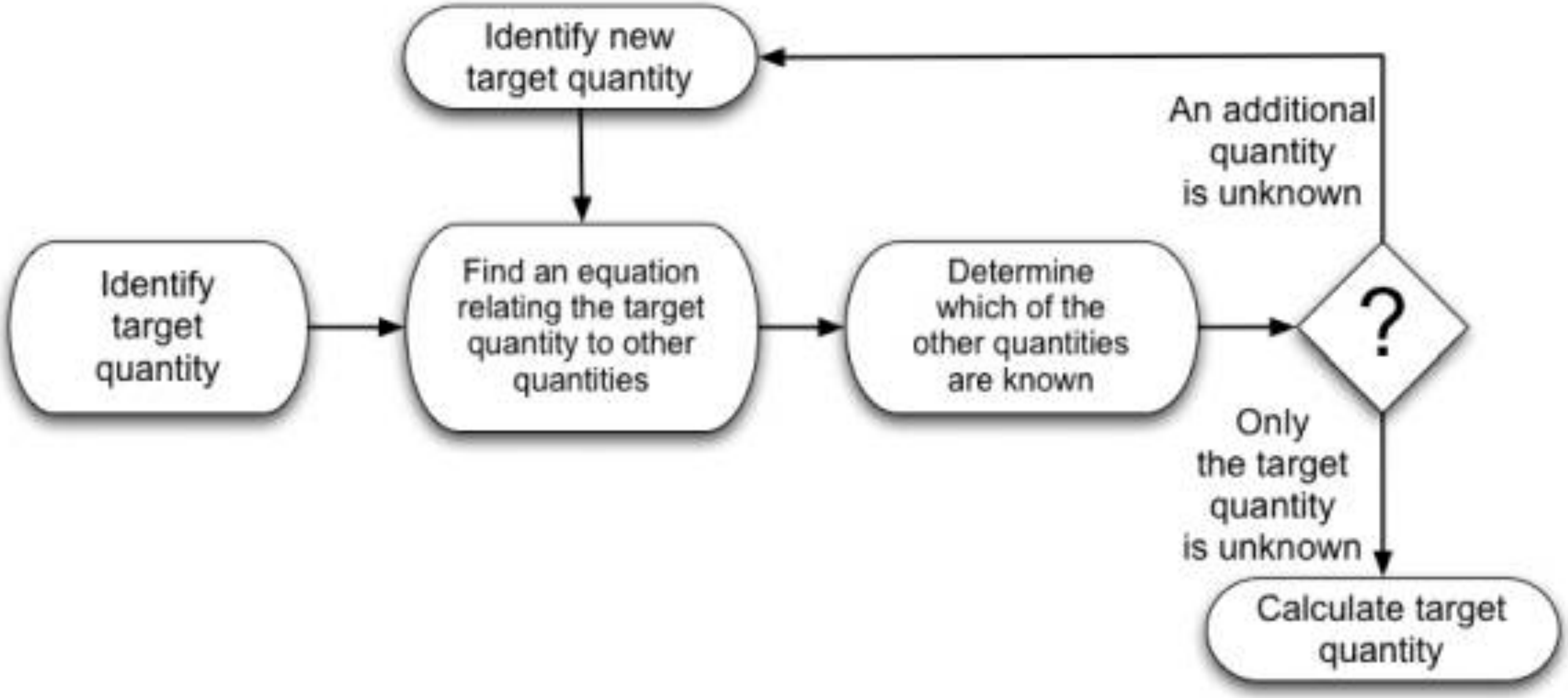
- For physicists, math is the “go to” epistemological resource – the one activated first and the one brought in to support intuitions and results developed in other ways.
- For biology students, the math is decidedly secondary. Teleology (structure/function) tends to be the “go to” resource.
- Part of our goal in teaching physics to second year biologists is to improve their understanding of the potential value of mathematical modeling. This means **teaching it** rather than assuming it.

+ A potentially useful tool: Epistemic games

- ***Epistemic game:*** A structured activity usable for approaching a variety of knowledge building tasks and problems.
  - Has an entry point, rules, an end point
  - Making a list
  - Compare and contrast
  - Cost-benefit analysis
  - Mechanism analysis (time, space, relationships)
  - Recursive plug-and-chug

Collins & Ferguson, *Educ. Psychol*. 28 (1993) 25
Bing & Redish, *Phys. Rev. ST-PER 5* (2009) 020108;
Bing & Redish, *Phys. Rev. ST-PER 8* (2012) 010105
Tuminaro & Redish, *Phys. Rev. ST-PER 3* (2007) 020101.

Recursive plug-and-chug



+ Example 3: Filling in the missing tools.

When a small organism is moving through a fluid, it experiences both viscous and inertial drag. The viscous drag is proportional to the speed and the inertial drag to the square of the speed. For small spherical objects, the magnitudes of these two forces are given by the following equations:

$$F_v = 6\pi\mu R v$$

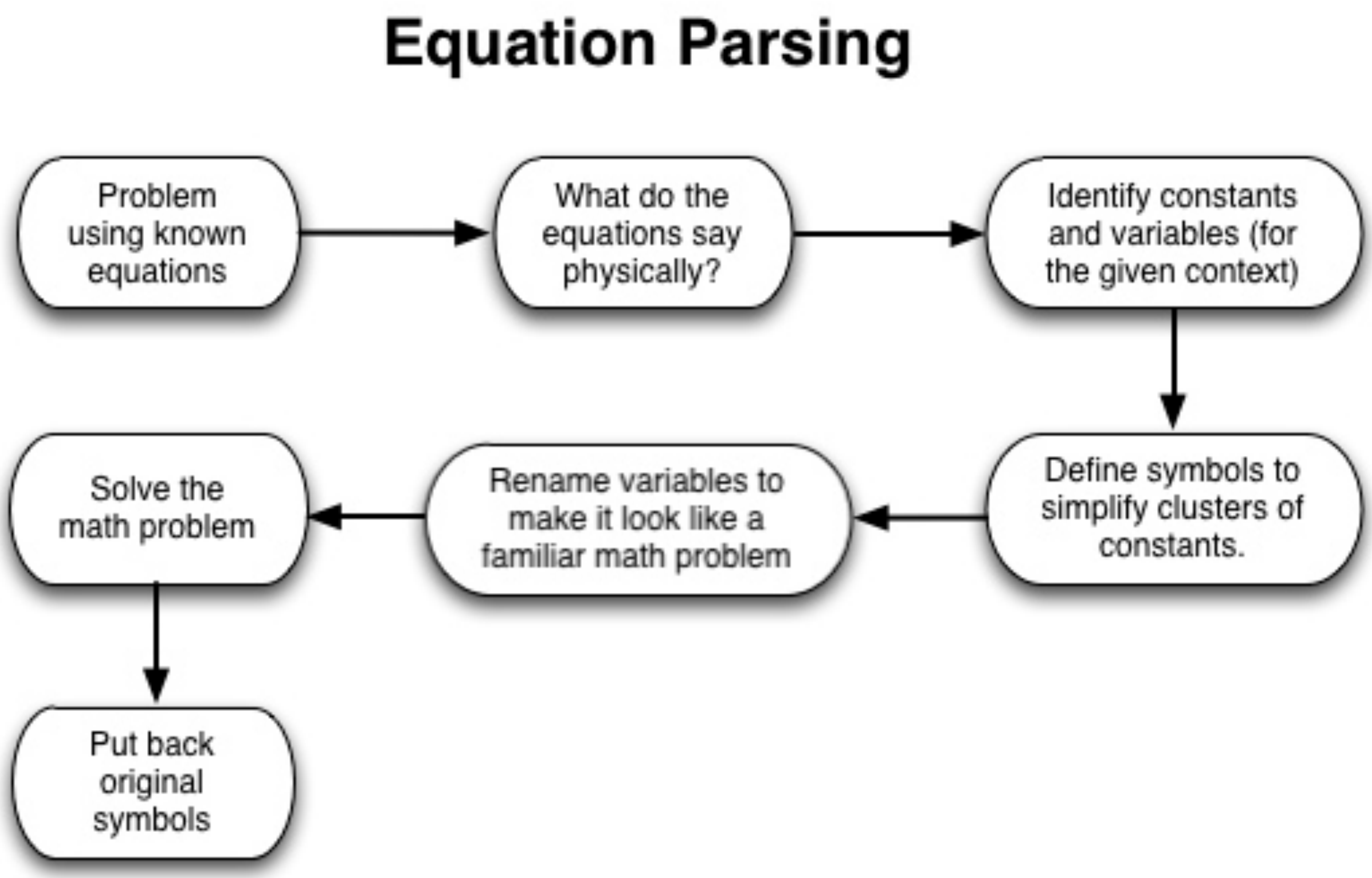
$$F_i = C\rho R^2 v^2$$

For a given organism (of radius *R*) is there ever a speed for which these two forces have the same magnitude?

+ Many students were seriously confused and didn't know what to do next.

- "Should I see if I can find all the numbers on the web?"
- "I don't know how to start."
  - *"Well, it says 'Do they ever have the same magnitude?' How do you think you ought to start?*
- "Set them equal?"
  - "OK. Do it."
- "I don't know what all these symbols mean."
  - "*Well everything except the velocity are constants for a particular object in a particular situation."*
- "Oh! So if I write it .... $Av = Bv^2$... Wow! Then it's easy!"

A useful epistemic game*



+ Conclusion

- It is clear from these varied analytical threads that doing math in science is not a simple "transfer" of what was learned in math to a science class.
- Understanding the different ways that meaning can be put to symbol may potentially help us both understand
  - Why students have so much trouble using math in science.
  - Why faculty have so much trouble empathizing with them.

Redish & Kuo, *Science & Educ.* 77 (2015) doi10.1007/s11191-015-9749-7, 30 pp.